Deductive Puzzling

Students can improve their deductive reasoning and communication skills by working on number puzzles.

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> The Martin family has four children—all of different ages. Phillip is older than Lynette. Carl is younger than Phillip. Diana is older than Phillip and Lynette. Carl is not the youngest. Who is the oldest?

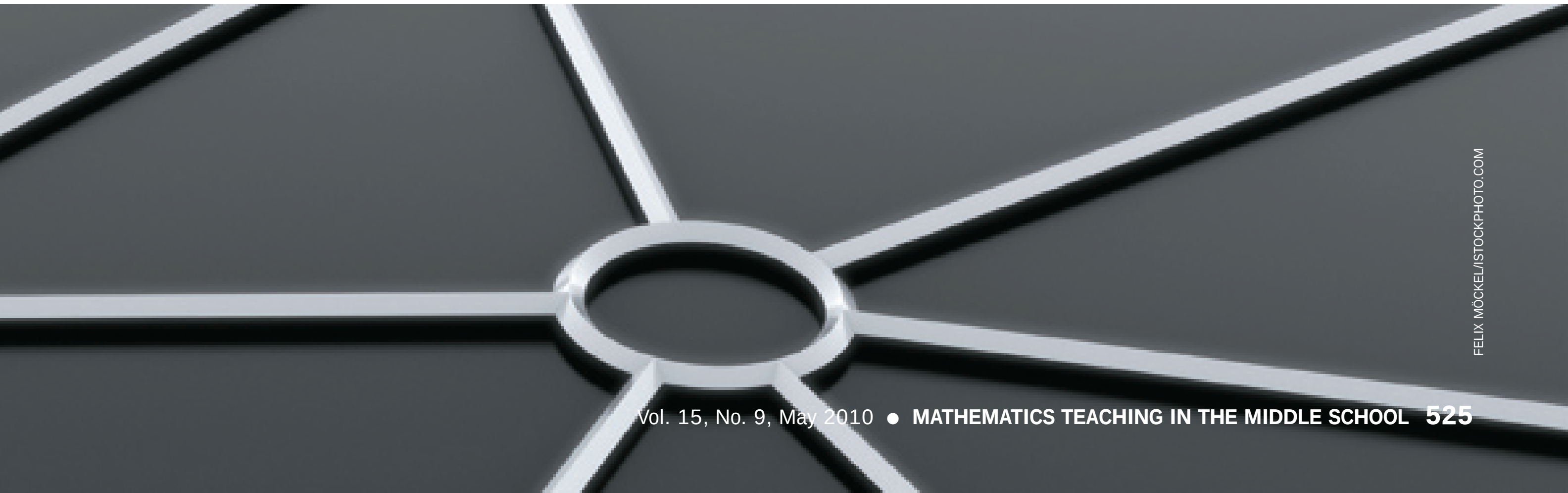
Would your students know where to begin answering this question? Would they draw a diagram to help sort out the information? Or would they become overwhelmed by the possibilities and be unable to respond?

To help fifth- through eighth-grade students develop their deductive reasoning skills, I used a ten-week supplementary curriculum so that students could answer logic questions similar to this one. The curriculum, a series of lessons built around language-independent logic puzzles, has been used in classrooms of fifth through eighth grades. In most cases, students' deductive-reasoning skills have increased, as measured by a Logical Thinking Inventory, a set of twenty logic questions including some like that above. Students have shown great interest in solving puzzles, discussing their problem-solving strategies, and applying what they have learned to other logic problems.

The curriculum is built around five different types of puzzles, but only two—*Shikaku* and *Hashiwokakero*—will be the focus here. Students' discoveries and strategies as well as how these puzzles support the math that is important for middle school students will be explored.

BACKGROUND

NCTM elevated the awareness of proof and proving by expanding the Process Standard of Reasoning that was found in *Curriculum and Evaluation Standards* (1989) to Reasoning and Proof, as found in *Principles and Standards* (2000). *Proof* can be viewed as a formal expression of reasoning that is a critical step in the analytical process (NCTM 2000). Unfortunately, the concept of proving something mathematically is often relegated to one high school course and usually within geometry, a single field of mathematics. This may be one reason why so many students struggle with proofs and deductive reasoning (Moore 1994) and why "the learning of proof and proving in school



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mathematics should be developmental and should start in the early grades" (ICMI 2008).

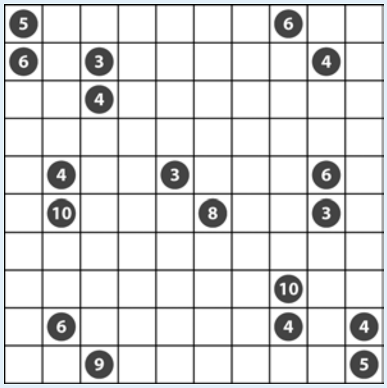
Deductive reasoning forms the core of most rigorous proofs. It is one place to start when working with younger students. *Deduction* can be thought of as the process of determining what has to be true (or what cannot be true) on the basis of what is already known. Language-independent logic puzzles come in many different varieties (e.g., Sudoku, Kakuro, Shikaku, Nurikabe, and Hashiwokakero), but their solutions all depend on applying deductive reasoning. They are language-independent in that knowledge of a language or culture does not affect someone's ability to solve the puzzle once the general goal and rules are known, unlike crossword and other word puzzles.

Over the ten-week period, I met with a class of academically gifted fifth graders during one forty-minute class each week. Together we worked on some of these puzzles. In other classes, the curriculum materials were taught to students by their teachers. In almost every case, each of the puzzle types used was new to all the students (I did not use Sudoku puzzles), so the strategies for solving them had to be developed as we went. I, or the regular classroom teachers, typically introduced a new type of puzzle one week, and students would collaboratively solve several sample puzzles. They were also given more puzzles to work on for homework. In class the following week, we would revisit those puzzles that caused students to struggle. We also analyzed and discussed some solution strategies. Students were then given more difficult or larger puzzles.

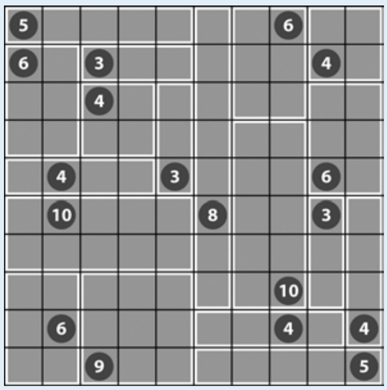
SHIKAKU

Students were shown an example of Shikaku as it would initially appear, together with a completed solution (see **fig. 1**). They were asked to describe what they thought the goal and rules for solving this type of puzzle would be.

Fig. 1 The goal of a Shikaku puzzle is to generate rectangles whose area is the same as the number encompassed in each rectangle. To begin, students were given a puzzle and its solution.

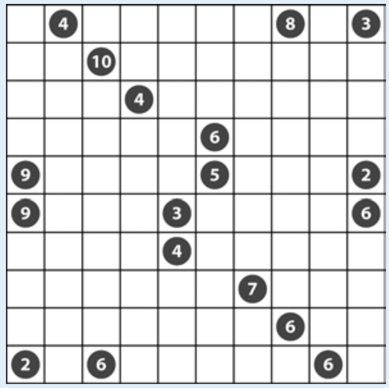


(a)
Shikaku example



(b)
Shikaku solution

Fig. 2 After more practice, students were given a second Shikaku puzzle to solve.



Initially, one student guessed that the idea was to "divide up the [large] square into regions so that the areas matched the given numbers." Another student wondered whether the regions needed to be "squares and rectangles." I explained that the Japanese word *Shikaku* means "four corners" or "rectangle." Students recognized the goal of subdividing the grid into rectangular regions along the grid lines, such that the area of each rectangle is given by the number encompassed in the rectangle. It helped when students acknowledged that a square is a kind of rectangle. I then introduced these additional rules:

- Only one number can appear in each rectangle.
- Each square on the grid is used in exactly one rectangle (no rectangles may overlap).

Once the goal and rules were established, students were given a new puzzle (see **fig. 2**) and were asked to work in pairs to find a solution. They were also asked to write a response to the prompt, "What are some things to look for as starting points in Shikaku puzzles?"

After all pairs had found the solution, we gathered ideas about how to start the puzzle. One student suggested "looking for rectangles that can only go one way." When asked to elaborate, he said, "You need to find a number that has only one possible rectangle with that area that covers that number." I asked for an example, and another student mentioned the 3 in the upper-right corner: "The only rectangle with an area of 3 is a 1 by 3 that starts at the corner and goes down."

At that point, another student pointed out that the 3 is special because there are few rectangles with an area of 3, just a 3×1 and a 1×3. Several students quickly said that 3 is prime and that prime numbers might also be good starting points for solving a Shikaku puzzle.

Finally, a student suggested looking at “numbers like 9.” In explanation, she said that “9 could be 1×9 or 9×1, but it could also only be 3×3—a square.” We discovered that, in fact, the two 9s on the left side of the puzzle could each only be covered one way, each by a 3×3 square. We discussed the fact that numbers that are the square of a prime number, p, would only have three possible associated rectangles: $1 \times p^2$; $p^2 \times 1$; and $p \times p$.

This discussion of solution strategies was enlightening and exciting. Students were interested in finding new and different ways to solve the puzzles. We generated a list of possible strategies and explored additional puzzles to see if these strategies would hold. We investigated whether other strategies would arise in new situations. Students were also building on their understanding of prime numbers, factors, factor pairs, and multiplication facts. They were encouraged to use this vocabulary when discussing their solutions and strategies.

Each puzzle type was introduced in a similar way, with students developing solution strategies and sharing them with the class. Students were always asked to write their explanations in words and to show examples of their strategies. Our recap of rules and strategies was followed by some puzzles that would require some different approaches to find a solution. **Activity sheet 1** contains four Shikaku puzzles that students can solve.

HASHIWOKAKERO

Hashiwokakero puzzles, or Hashi for short, have more rules to follow than Shikaku puzzles. However, once the rules are understood, students quickly develop problem-solving strategies, especially after they learn that *Hashiwokakero* in Japanese means “build bridges.”

The starting grid for a Hashi puzzle consists of a series of circles that represent islands (see **fig. 3a**). Each circle contains a number from 1 to 8. The goal is to connect all the islands with bridges, so that each island can be reached from any other island in the grid. These additional bridge rules must be followed:

- Bridges must begin and end at islands.
- Travel must be done in a horizontal or vertical line only.
- Bridges must not cross any other bridges or islands.
- No more than two bridges can connect a pair of islands.
- The total number of bridges connected to each island must match the number on that island.

We looked at an example of a Hashi starting grid and its corresponding solution (see **fig. 3**). After articulating the goal and the rules, students verified that each rule was followed and that all islands were connected. I then challenged the students to explore some additional Hashi puzzles and identify some initial bridges that could be drawn. As with the previous puzzles, this prompt engaged the students in finding ways to start solving the puzzle. It also helped them identify some problem-solving strategies that could be applied to other Hashi puzzles.

With a new unsolved Hashi puzzle (see **fig. 4**), students discussed some strategies that showed their developing deductive-reasoning skills and their ability to generalize situations from a specific example. When I asked where they could start, students could not immediately answer. They had seen a Hashi puzzle with a 4 in a corner and they knew that, in that case, two

Reflect and Discuss

Using logic puzzles is a great way to have students concentrate not only on problem solving but also on reasoning habits. The lessons offered here illustrate how students connect to previous learning (discussing primes and squares), communicate with one another (brainstorming and combining ideas), use representation (choosing geometric models of the patterns that occur with prime and square numbers), and generalize their ideas to create methods for solving the puzzles. This process is helping students extend their thinking and reflect on their solving processes.

Some specific habits of reasoning occur when students work through Shikaku and Hashiwokakero (Hashi) puzzles, described in this article. Students look for hidden structures as they analyze how different rectangles can be drawn and fit together in the problems involving Shikaku puzzles. When deciding how many bridges can be made between two numbers in the Hashi puzzle problems, students are making conjectures and deductions to see what is feasible. While using both puzzles, students revisit their initial assumptions and reconcile different approaches when the class generalizes solution methods. Communicating mathematical ideas is another important aspect of reasoning and sense making that these lessons emphasize.

The puzzles also help teachers develop the reasoning habits of their students by providing tasks that require them to problem solve for themselves. Teachers model important pedagogy by allowing students to have time to analyze the problem, by not telling students what steps may be useful, and by having students justify their reasoning.

bridges would be in each of the two possible directions. In this puzzle, a 3 in the top-right corner meant that one bridge could be to the left and two down, or two to the left and one down. Finally, one student said,

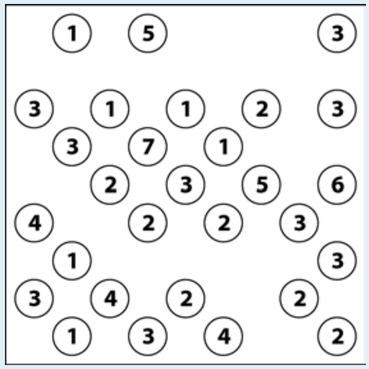
> I'm not sure if this helps, but since there has to be *at least* one bridge to the left and one down, we could draw those two (one to the left and one down), and then add the third one when we know for sure.

Students agreed that her reasoning was correct. I explained that Hashi solvers sometimes shade in a circled number when all the bridges have been drawn from that island, which indicates whether he or she still needs to complete the requirements for that island.

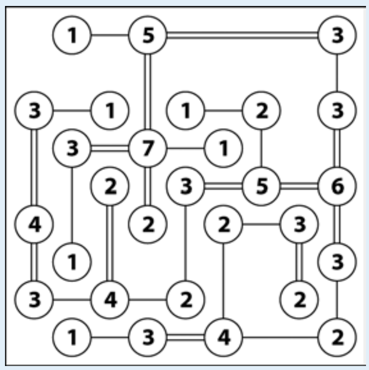
I asked the class whether any other islands could determine the placement of all or some but not all the bridges to that island. Eventually, students found several other situations that required drawing one bridge in each of the possible directions, although not yet satisfying all the requirements for the island, such as when a 5 appears along an edge and a 7 appears in the middle of the grid. Each of these generalities grew from a specific situation from this and other puzzles and showed the students' reasoning about the minimum number of bridges that could be drawn in various directions. Students also found that some of these moves were helpful when starting to work on a new Hashi puzzle.

One other strategy that arose from this puzzle occurs in the top-left corner. Students learned to look at the corners of Hashi puzzles for some potential starting bridges. One student quickly noted that he could not draw a bridge between the two 1s in the top-left corner, since that would isolate those two islands from the rest of the ones in the grid. Later, another student noticed that any two islands labeled with 2s could not be connected with two bridges between them for the same reason. This strategy can be very useful in the corners and in some of the larger Hashi puzzles. **Activity sheet 2** contains four Hashi puzzles that students can solve.

Fig. 3 Using only horizontal and vertical bridges, the goal of Hashiwokakero puzzles is to connect all the islands, so that each island can be reached from any other island in the grid.



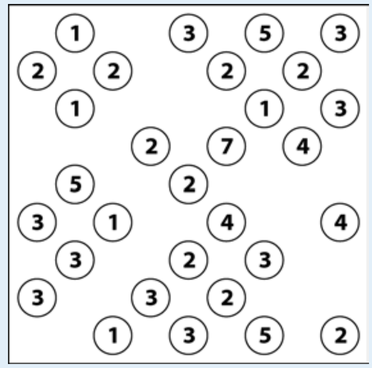
(a)
Hashiwokakero example



(b)
Hashiwokakero example solution

For the remainder of the lesson, three additional puzzle types were introduced to the students—dominoes, Masyu, and Kakuro. Each puzzle had a unique solution, and students were always prompted to describe their solution strategies. Students developed language that was typically used when they discussed their strategies: "I know *this has* to be true because I already know . . ." or "Because *this* and *this* don't work, this *has* to be true."

Fig. 4 After verifying a solution to a given Hashiwokakero puzzle, students were given a second puzzle to solve.



It became clear that they were becoming comfortable applying the discourse of deductive reasoning to the statements of their solutions. I was also interested in finding out whether they would be able to apply this kind of mathematical thinking to the questions found on the Logical Thinking Inventory.

Recognizing that the test itself is not language-independent, that this class of students was not typical, and that there was no control group for comparison, I continued to wonder whether much growth would be found in the students' ability to answer logic questions. At the conclusion of the curriculum, students took the same Logical Thinking Inventory that they had been given as a pretest, and their scores were compared. The mean score for the class increased from 68 percent to 77 percent. On a test of twenty questions, seven out of twenty students improved their score by three or more additional correct answers. These results indicate that there is potential for using a supplemental puzzle curriculum to help students develop their deductive-reasoning skills.

NEXT STEPS

This pilot study was limited in scope and types of subjects. A more thorough study including a larger number of students in different classroom settings would be helpful in establishing a more significant impact. It would also be helpful to compare the work of students who are using the puzzle curriculum with those who are not.

With this class, however, I found that after spending ten weeks working on logic puzzles and discussing the underlying deductive reasoning used to solve them, students were better at working on basic logic tasks. Students not only correctly answered more questions on the Logical Thinking Inventory but also attempted to answer more questions in general, which demonstrated their increased confidence in delving into logic problems. They also drew more diagrams and organized their thinking in more structured and sensible ways.

My involvement with language-independent logic puzzles began as a recreational pastime. It has since become an integral part of my work with preservice and in-service teachers. It is also a line of research that I find fascinating. I believe that puzzles such as these could play a critical role in impacting the development of our students' logical-reasoning skills.

REFERENCES

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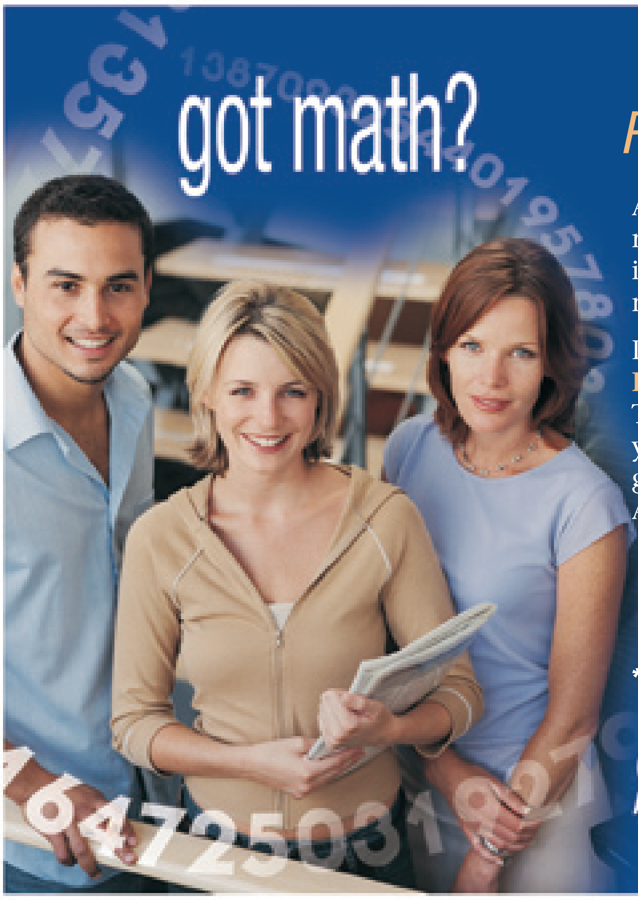
Ed. note: For more logic puzzles, read "Japanese Logic Puzzles and Proof," also by Jeffrey J. Wanko, in *Mathematics Teacher* (Nov. 2009, pp. 266–71).

The solutions to **activity sheets 1** and **2** are appended to the online version of this article at **www.nctm.org/mtms**.



Name ______________________________

SHIKAKU

The rules:

1. Each of the following Shikaku puzzles needs to be sectioned into rectangles (and squares) along the grid lines, so that the number in each rectangle refers to the area of that rectangle.

2. Only one number can appear in each rectangle.

3. Each square on the grid is used in exactly one rectangle (that is, no rectangles may overlap).

			8	2					4
				4				6	8
		8							
		8	4						
						3	6		
							9		
5	3				12				
2					4	4			

Puzzle 1

						3		3	
		8		2					4
	6								
						10			2
	4			6					
					8			6	
5			4						
								9	
5					2		6		
	3		4						

Puzzle 2

						10			
				6					
		6					3		
5									
				3	3			6	
	6			5	4				
									8
		12					10		
					6				
			7						

Puzzle 3

	4	3			5				
				4				6	
3	5								4
		8				6			
			10				8		
3								8	5
	2				4				
				6			4	2	

Puzzle 4